PERFECT FLUID AND TEST PARTICLE WITH SPIN AND DILATONIC CHARGE IN A WEYL–CARTAN SPACE

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Abstract

The equation of perfect dilaton-spin fluid motion in the form of generalized hydrodynamic Euler-type equation in a Weyl–Cartan space is derived. The equation of motion of a test particle with spin and dilatonic charge in the Weyl–Cartan geometry background is obtained. The peculiarities of test particle motion in a Weyl–Cartan space are discussed.

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1. INTRODUCTION

In a previous work[1] (which we shall refer as I) the variational theory of the perfect fluid with intrinsic spin and dilatonic charge (dilaton-spin fluid) was developed and the equations of motion of this fluid, the Weyssenhoff-type evolution equation of the spin tensor and the conservation law of the dilatonic charge were derived.

The purpose of the present work is to investigate the equations of motion of such type of fluid and their consequences, one of which leads to the equation of motion of a test particle with spin and dilatonic charge in the Weyl–Cartan background.

It is well known that the equations of charge particle motion in an electromagnetic theory are the consequence of the covariant energy-momentum conservation law of the system 'particles–field' and the electromagnetic field equations.[2] In General Relativity the equations of matter motion are the consequence of the gravitational field equations. The reason consists in the fact that the Einstein equations lead to the covariant energy-momentum conservation law of matter. In the Einstein–Cartan theory[3,4] the same situation occurs, citeHe:let but the conservation laws have more complicated form established in Ref. 4. In Refs. 6, 7 it was proved that in the generalized theories of gravity with torsion in a Riemann–Cartan space U_4 based on non-linear Lagrangians the equations of the matter motion are also the consequence of the gravitational field equations. The similar result was established in a metric-affine space with curvature, torsion and nonmetricity.[8−10]

In Sec. 2 we shall use this method for deriving the equation of dilaton-spin fluid motion in the form of generalized hydrodynamic Euler-type equation in a Weyl–Cartan space. In Sec. 3 this equation will be applied for obtaining the equation of motion of a test particle with spin and dilatonic charge in the Weyl–Cartan geometry background.

2. THE HYDRODYNAMIC EQUATION OF MOTION OF THE PERFECT DILATON-SPIN FLUID

In a Weyl–Cartan space Y_4 the matter Lagrangian obeys the diffeomorfism invariance, the local Lorentz invariance and the local scale invariance that leads to the corresponding Noether identities which can be obtained as the particular case of the corresponding identities stated in a general metric-affine space[11] (see Appendix on the notations used),

$$\mathcal{D}\Sigma_\sigma = (\bar{e}_\sigma \rfloor \mathcal{T}^\alpha) \wedge \Sigma_\alpha - (\bar{e}_\sigma \rfloor \mathcal{R}^\alpha{}_\beta) \wedge \mathcal{J}^\beta{}_\alpha - \frac{1}{8}(\bar{e}_\sigma \rfloor \mathcal{Q})\sigma^\alpha{}_\alpha \ , \tag{2.1}$$

$$\left(\mathcal{D} + \frac{1}{4}\mathcal{Q}\right) \wedge \mathcal{S}_{\alpha\beta} = \theta_{[\alpha} \wedge \Sigma_{\beta]} \ , \tag{2.2}$$

$$\mathcal{D}\mathcal{J} = \theta^\alpha \wedge \Sigma_\alpha - \sigma^\alpha{}_\alpha \ . \tag{2.3}$$

Here Σ_σ is the canonical energy-momentum 3-form, $\sigma_{\alpha\beta}$ is the metric stress-energy 4-form, $\mathcal{S}_{\alpha\beta}$ is the spin momentum 3-form and $\mathcal{J}$ is the dilaton current 3-form.

In case of the perfect dilaton-spin fluid the corresponding expressions for the quantities Σ_σ, $\sigma_{\alpha\beta}$, $\mathcal{S}_{\alpha\beta}$ and $\mathcal{J}$ were derived in I (see (I.5.3), (I.5.5), (I.5.6)). These expressions are compatible in the sense that they satisfy to the Noether identities (2.1), (2.2) and (2.3). The identities (2.2) and (2.3) can be verified with the help of the spin tensor evolution equation (I.4.4) and the dilatonic charge conservation law (I.4.2).

The Noether identity (2.1) represents the quasiconservation law for the canonical matter energy-momentum 3-form. This identity is fulfilled, if the equations of matter motion are valid, and therefore represents in its essence another form of the matter motion equations.

Let us introduce with the help of (I.5.4) a specific (per particle) dynamical momentum of a fluid element,

$$\pi_\sigma \eta := -\frac{1}{nc^2} *u \wedge \Sigma_\sigma \ , \quad \pi_\sigma = \frac{\varepsilon}{nc^2}u_\sigma - \frac{1}{c^2}S_{\sigma\rho}\bar{u}\rfloor \mathcal{D}u^\rho \ . \tag{2.4}$$

Then the canonical energy-momentum 3-form (I.5.4) reads,

$$\Sigma_\sigma = p\eta_\sigma + n\left(\pi_\sigma + \frac{p}{nc^2}u_\sigma\right)u \ . \tag{2.5}$$

Substituting (2.5), (I.5.5) and (I.5.6) into (2.1), one obtains after some algebra the equation of motion of the perfect dilaton-spin fluid in the form of the generalized hydrodynamic Euler-type equation of the perfect fluid,

$$
\begin{aligned}
&u \wedge \mathcal{D}\left(\pi_\sigma + \frac{p}{nc^2}u_\sigma\right) = \frac{1}{n}\eta\bar{e}_\sigma\rfloor\mathcal{D}p - \frac{1}{8n}\eta(\varepsilon+p)Q_\sigma \\
&-(\bar{e}_\sigma\rfloor\mathcal{T}^\alpha)\wedge\left(\pi_\alpha + \frac{p}{nc^2}u_\alpha\right)u - \frac{1}{2}(\bar{e}_\sigma\rfloor\mathcal{R}^{\alpha\beta})\wedge S_{\alpha\beta}u + \frac{1}{8}(\bar{e}_\sigma\rfloor\mathcal{R}^\alpha{}_\alpha)\wedge Ju\ .
\end{aligned}
\tag{2.6}
$$

Let us evaluate the component of the equation (2.6) along the 4-velocity by contracting one with u^σ. After some algebra we get the energy conservation law along a streamline of the fluid,

$$
d\varepsilon = \frac{\varepsilon+p}{n}dn\ . \tag{2.7}
$$

Comparing this equation with the first thermodynamic principle (I.2.14), one can conclude that along a streamline of the fluid the conditions

$$
ds = 0\ , \qquad \frac{\partial\varepsilon}{\partial S^p{}_q}dS^p{}_q = 0\ , \qquad \frac{\partial\varepsilon}{\partial J}dJ = 0\ . \tag{2.8}
$$

are valid. The first of these equalities means that the entropy conservation law is fulfilled along a streamline of the fluid. This fact corresponds to the basic postulates of the theory.

3. THE EQUATION OF TEST PARTICLE MOTION IN A WEYL–CARTAN SPACE

Let us consider the limiting case when the pressure p vanishes, then the equation (2.6) will describe the motion of one fluid particle with the mass $m_0 = \varepsilon/(nc^2) = const$, with the spin tensor $S_{\alpha\beta}$ and the dilatonic charge J,

$$
u\wedge\mathcal{D}\pi_\sigma = -\frac{1}{8}\eta m_0 c^2 Q_\sigma - (\bar{e}_\sigma\rfloor\mathcal{T}^\alpha)\wedge\pi_\alpha u - \frac{1}{2}(\bar{e}_\sigma\rfloor\mathcal{R}^{\alpha\beta})\wedge S_{\alpha\beta}u + \frac{1}{8}(\bar{e}_\sigma\rfloor\mathcal{R}^\alpha{}_\alpha)\wedge Ju\ . \tag{3.1}
$$

The third term on the right-hand side of (3.1) represents the well-known Mathisson force, the second term represents the translational force that appears in spaces with torsion. The

forth term appears only in a Weyl–Cartan space. It has the Lorentz-like form with the Weyl's homothetic curvature tensor $\mathcal{R}^{\alpha}{}_{\alpha}$ as dilatonic field strength.

The following Theorem is valid.

Theorem. In a Weyl–Cartan space Y_4 the motion of a test particle with spin and dilatonic charge obeys the equation,

$$m_0 u \wedge \overset{R}{\mathcal{D}}\, u_\sigma = \frac{1}{c^2} u \wedge \left(\delta^\alpha_\sigma \overset{C}{\mathcal{D}} - \bar{e}_\sigma \rfloor \mathcal{T}^\alpha \right) * u^\beta \overset{C}{\mathcal{D}}\, (S_{\alpha\beta} u)$$
$$- \frac{1}{2} (\bar{e}_\sigma \rfloor \overset{C}{\mathcal{R}}{}^{\alpha\beta}) \wedge S_{\alpha\beta} u + \frac{1}{16} (\bar{e}_\sigma \rfloor d\mathcal{Q}) \wedge J u \ , \tag{3.2}$$

where $\overset{C}{\mathcal{R}}{}^{\alpha\beta}$ is a Riemann–Cartan curvature 2-form, $\overset{R}{\mathcal{D}}$ and $\overset{C}{\mathcal{D}}$ are the exterior covariant differentials with respect to a Riemann connection 1-form $\overset{R}{\Gamma}{}^{\alpha}{}_{\beta}$ and a Riemann–Cartan connection 1-form $\overset{C}{\Gamma}{}^{\alpha}{}_{\beta}$, respectively.

Proof. Using the decomposition (A.5) (see Appendix) the specific dynamical momentum of a fluid element (2.4) can be written in the form,

$$\pi_\sigma = m_0 u_\sigma - \frac{1}{c^2} S_{\sigma\rho} \bar{u} \rfloor \overset{C}{\mathcal{D}}\, u^\rho - \frac{1}{8} S_{\sigma\rho} Q^\rho \ . \tag{3.3}$$

With the help of the decomposition (A.5) one can prove that the evolution equation of the spin tensor (I.4.4) is also valid with respect to the Riemann–Cartan connection $\overset{C}{\Gamma}{}^{\alpha}{}_{\beta}$ and reads,

$$\Pi^\alpha_\sigma \Pi^\rho_\beta\, \bar{u} \rfloor \overset{C}{\mathcal{D}}\, S^\sigma{}_\rho = 0 \ , \tag{3.4}$$

where $\Pi^\alpha_\sigma := \delta^\alpha_\sigma + c^{-2} u^\alpha u_\sigma$ is the projection tensor. Using (2.4) and (3.4) the left-hand side of the equation (3.1) can be represented as follows,

$$u \wedge \mathcal{D} \pi_\sigma = m_0 u \wedge \overset{C}{\mathcal{D}}\, u_\sigma - \frac{1}{8} \eta m_0 c^2 Q_\sigma - \frac{1}{c^2} u \wedge \overset{C}{\mathcal{D}}\, (S_{\sigma\rho} \bar{u} \rfloor \overset{C}{\mathcal{D}}\, u^\rho)$$
$$- \frac{1}{8} S_{\sigma\rho} u \wedge \overset{C}{\mathcal{D}}\, Q^\rho - \frac{1}{64} \eta S_{\sigma\rho} Q^\rho Q_\lambda u^\lambda \ . \tag{3.5}$$

With the help of the decomposition (A.6) the third term on the right-hand side of (3.1) takes the form,

$$
\frac{1}{2}(\bar{e}_\sigma\rfloor\mathcal{R}^{\alpha\beta})\wedge S_{\alpha\beta}u=\frac{1}{2}(\bar{e}_\sigma\rfloor\overset{C}{\mathcal{R}}{}^{\alpha\beta})\wedge S_{\alpha\beta}u+\frac{1}{8}\eta T^{\alpha}{}_{\sigma\rho}u^{\rho}S_{\alpha\beta}Q^{\beta}
-\frac{1}{8}\eta S_{\sigma\rho}\bar{u}\rfloor\overset{C}{\mathcal{D}}\,Q^{\rho}+\frac{1}{64}\eta S_{\sigma\rho}Q^{\rho}Q_{\lambda}u^{\lambda}\,. \tag{3.6}
$$

Substituting (3.6) and (3.3) in the right-hand side of (3.1), comparing the result with (3.5) and taking into account the equality (A.12), one obtains (3.2), as was to be proved.

The Theorem proved has the important consequences.

Corollary 1. The motion of a test particle without spin and dilatonic charge in a Weyl–Cartan space Y_4 coincides with the motion of this particle in the Riemann space, the metric tensor of which coincides with the metric tensor of Y_4.

Corollary 2. The motion of a test particle with spin and dilatonic charge in a Weyl–Cartan space Y_4 coincides with the motion of this particle in the Riemann–Cartan space, the metric tensor and the torsion tensor of which coincide with the metric tensor and the torsion tensor of Y_4, if one of the conditions is fulfilled:

i) the dilatonic field is a closed form, $d\mathcal{Q}=0$;

ii) the dilatonic charge of the particle vanishes, $J=0$.

Corollary 3. The manifestations of the non-trivial Weyl space structure (when the dilatonic field $\mathcal{Q}$ is not a closed form) can be detected only with the help of the test particle endowed with dilatonic charge.

The result of the Corollary 1 can be cosidered as a particular case of the Theorem stated in Refs. 8 – 10 for the matter motion in a general metric-affine space.

4. CONCLUSIONS

The perfect dilaton-spin fluid model represents the medium with spin and dilatonic charge which generates the spacetime Weyl–Cartan geometrical structure and interacts with it. The influence of the Weyl–Cartan geometry on dilaton-spin fluid motion is described by the Euler-type hydrodynamic equation. This hydrodynamic equation leads to the equation of motion of a test particle with spin and dilatonic charge in the Weyl–Cartan geometry

background, the special form of which is stated by the Theorem of Sec. 3.

The important consequences of this Theorem mean that bodies and mediums without dilatonic charge are not subjected to the influence of the possible Weyl structure of spacetime (in contrast to the generally accepted opinion) and therefore can not be the tools for the detection of the Weyl properties of spacetime. For 'usual' matter without dilatonic charge the Weyl structure of spacetime is unobservable. In order to investigate the different manifestations of the possible Weyl structure of spacetime one needs to use the bodies and mediums endowed with dilatonic charge.

APPENDIX:

Let us consider a connected 4-dimensional oriented differentiable manifold $\mathcal{M}$ equipped with a linear connection Γ and a metric g of index 1, which are not compatible in general in the sense that the covariant exterior differential of the metric does not vanish,

$$\mathcal{D}g_{\alpha\beta} = dg_{\alpha\beta} - \Gamma^{\gamma}{}_{\alpha}g_{\gamma\beta} - \Gamma^{\gamma}{}_{\beta}g_{\alpha\gamma} =: -\mathcal{Q}_{\alpha\beta}\ , \tag{A.1}$$

where $\Gamma^{\alpha}{}_{\beta}$ is a connection 1-form and $\mathcal{Q}_{\alpha\beta}$ is a nonmetricity 1-form, $\mathcal{Q}_{\alpha\beta} = Q_{\alpha\beta\gamma}\theta^{\gamma}$.

A curvature 2-form $\mathcal{R}^{\alpha}{}_{\beta}$ and a torsion 2-form $\mathcal{T}^{\alpha}$,

$$\mathcal{R}^{\alpha}{}_{\beta} = \frac{1}{2}R^{\alpha}{}_{\beta\gamma\lambda}\theta^{\gamma}\wedge\theta^{\lambda}\ , \qquad \mathcal{T}^{\alpha} = \frac{1}{2}T^{\alpha}{}_{\beta\gamma}\theta^{\beta}\wedge\theta^{\gamma}\ , \tag{A.2}$$

are defined by virtue of the Cartan's structure equations,

$$\mathcal{R}^{\alpha}{}_{\beta} = d\Gamma^{\alpha}{}_{\beta} + \Gamma^{\alpha}{}_{\gamma}\wedge\Gamma^{\gamma}{}_{\beta}\ , \qquad \mathcal{T}^{\alpha} = \mathcal{D}\theta^{\alpha} = d\theta^{\alpha} + \Gamma^{\alpha}{}_{\beta}\wedge\theta^{\beta}\ . \tag{A.3}$$

A Weyl–Cartan space Y_4 is a space with a metric, curvature, torsion and nonmetricity which obeys the constraint ($\mathcal{Q}$ is a Weyl 1-form),

$$\mathcal{Q}_{\alpha\beta} = \frac{1}{4}g_{\alpha\beta}\mathcal{Q}\ , \qquad \mathcal{Q} := g^{\alpha\beta}\mathcal{Q}_{\alpha\beta} = Q_{\alpha}\theta^{\alpha}\ . \tag{A.4}$$

In a Weyl–Cartan space the following decomposition of the connection is valid,

$$\Gamma^{\alpha}{}_{\beta} = \overset{C}{\Gamma}{}^{\alpha}{}_{\beta} + \Delta^{\alpha}{}_{\beta} \ , \qquad \Delta^{\alpha}{}_{\beta} = \frac{1}{8}(2\theta^{[\alpha}Q_{\beta]} + \delta^{\alpha}_{\beta}\mathcal{Q}) \ , \tag{A.5}$$

where $\overset{C}{\Gamma}{}^{\alpha}{}_{\beta}$ denotes a connection 1-form of a Riemann–Cartan space U_4 with curvature, torsion and a metric compatible with a connection.

The decomposition (A.5) of the connection induces corresponding decomposition of the curvature,

$$\mathcal{R}^{\alpha}{}_{\beta} = \overset{C}{\mathcal{R}}{}^{\alpha}{}_{\beta} + \overset{C}{\mathcal{D}}\,\Delta^{\alpha}{}_{\beta} + \Delta^{\alpha}{}_{\gamma} \wedge \Delta^{\gamma}{}_{\beta} = \overset{C}{\mathcal{R}}{}^{\alpha}{}_{\beta} + \mathcal{P}^{\alpha}{}_{\beta} + \frac{1}{8}\delta^{\alpha}_{\beta} d\mathcal{Q} \ , \quad \mathcal{P}^{\alpha}{}_{\beta} = \mathcal{P}^{[\alpha}{}_{\beta]} \ , \tag{A.6}$$

$$\mathcal{P}^{\alpha}{}_{\beta} = \frac{1}{4}\left(\mathcal{T}^{[\alpha}Q_{\beta]} - \theta^{[\alpha} \wedge \overset{C}{\mathcal{D}}\,Q_{\beta]} + \frac{1}{8}\theta^{[\alpha}Q_{\beta]} \wedge \mathcal{Q} - \frac{1}{16}\theta^{\alpha} \wedge \theta_{\beta}Q_{\gamma}Q^{\gamma}\right) \ , \tag{A.7}$$

where $\overset{C}{\mathcal{D}}$ is the exterior covariant differential with respect to the Riemann–Cartan connection 1-form $\overset{C}{\Gamma}{}^{\alpha}{}_{\beta}$ and $\overset{C}{\mathcal{R}}{}^{\alpha}{}_{\beta}$ is the Riemann–Cartan curvature 2-form. In (A.6) the last term contains the Weyl homothetic curvature 2-form,

$$\mathcal{R}^{\alpha}{}_{\alpha} = \frac{1}{2}\mathcal{D}\mathcal{Q} = \frac{1}{2}(\bar{e}_{\alpha}\rfloor\mathcal{D}Q_{\beta})\theta^{\alpha} \wedge \theta^{\beta} + \frac{1}{2}Q_{\alpha}\mathcal{T}^{\alpha} = \frac{1}{2}d\mathcal{Q} \ . \tag{A.8}$$

The Riemann–Cartan connection 1-form can be decomposed as follows,

$$\overset{C}{\Gamma}{}^{\alpha}{}_{\beta} = \overset{R}{\Gamma}{}^{\alpha}{}_{\beta} + \mathcal{K}^{\alpha}{}_{\beta} \ , \quad \mathcal{T}^{\alpha} =: \mathcal{K}^{\alpha}{}_{\beta} \wedge \theta^{\beta} \ , \tag{A.9}$$

$$\mathcal{K}_{\alpha\beta} = 2\bar{e}_{[\alpha}\rfloor\mathcal{T}_{\beta]} - \frac{1}{2}\bar{e}_{\alpha}\rfloor\bar{e}_{\beta}\rfloor(\mathcal{T}_{\gamma} \wedge \theta^{\gamma}) \ , \tag{A.10}$$

where $\overset{R}{\Gamma}{}^{\alpha}{}_{\beta}$ is a Riemann (Levi–Civita) connection 1-form and $\mathcal{K}^{\alpha}{}_{\beta}$ is a kontorsion 1-form.[11] In a Riemann–Cartan space the covariant differentiation with respect to the transport connection[12] is useful,

$$\delta^{\rho}_{\sigma}\overset{tr}{\mathcal{D}} := \delta^{\rho}_{\sigma}\overset{C}{\mathcal{D}} - \bar{e}_{\sigma}\rfloor\mathcal{T}^{\rho} \ . \tag{A.11}$$

In particular, the following equality is valid,

$$u \wedge \overset{tr}{\mathcal{D}}\,u_{\sigma} = u \wedge \left(\delta^{\rho}_{\sigma}\overset{C}{\mathcal{D}} - \bar{e}_{\sigma}\rfloor\mathcal{T}^{\rho}\right)u_{\rho} = u \wedge \overset{R}{\mathcal{D}}\,u_{\sigma} \ . \tag{A.12}$$

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